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Ergodicity and the Lorentz transformation of time-based probabilities

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Abstract. One-particle systems and systems of non-interacting particles, free or confined to a container, are considered in this paper. The systems are assumed to have discrete states available to them, but the particles constituting them are considered as point particles. It is shown that a necessary condition for the ergodicity of a confined system to be a Lorentz-invariant concept is the introduction of a new kind of ensemble whose systems can be in motion but are *on average* at rest in one and the same inertial frame. The discrete probabilities of statistical mechanics are shown to lack Lorentz invariance (contrary to accepted ideas), and a transformation formula is derived for them.

1. Introduction

In equilibrium mechanics a usual procedure is to use ensembles of systems. Each system is a replica of the system under study and, though they are in different microscopic states and satisfy possibly different initial conditions, each system is subject to the same macroscopic conditions of constraint. These conditions may specify that the number of particles, the energy and the volume each lie in a narrow and prescribed range of values. Then the number of systems in the ensemble is assumed to be so large that the states of the system compatible with the constraints are well represented by the states of the systems of the ensemble. Hence one may average mechanical or other parameters of the system over the ensemble at any one time, and one will in this way arrive at average values of these parameters. These procedures are now well known, and some of the ensembles are well studied.

One need not think of the ensemble as a physical collection of systems or a fictitious collection of systems. It is quite adequate to fasten attention to the time-independent frequency or probability distribution to which an ensemble gives rise, while forgetting the collection of systems. Questions of the foundations of the concepts of probability theory, which are vaguely hinted at when speaking about statistical ensembles, might well be regarded as belonging to probability theory rather than statistical mechanics (Landsberg 1961, p. 388). In this paper, however, it is convenient to use the more picturesque ensemble language.

One can, alternatively, follow the behaviour of the equilibrium system under study for a long time and obtain the time average of the properties of interest. Which average yields the better representation of the thermodynamic properties of the system? The point is that the ensemble averages are easier to handle and much more frequently used; but one often seeks to justify them by proving them to be approximately equal to the time averages. This is the burden of the ergodic and quasi-ergodic hypotheses (e.g. Farquhar 1964), which are believed to be valid for most 'normal' systems.

Suppose one ergodic system and a corresponding ensemble are all at rest in an inertial frame I_0. In this paper we ask if the system is still ergodic for an observer in a general frame I. Two possibilities must then be considered. (*a*) One lays it down as a requirement that a single system and a single ensemble, applicable to all frames, must form the basis of the theorem. (*b*) Alternatively, one allows *one ensemble for each frame*. In case (*b*) the system under study will furnish certain time averages, and there is in principle no difficulty in setting up an ensemble with a frequency distribution which is in accord with time averages. In such a case the ergodic theorem clearly holds 'by construction', and the only problem left is to find an independent justification for the ensemble probability distribution by considering the nature of the energy shell of the system under study in the general frame I. It is clear, therefore, that case (*a*) raises the new points brought into ergodic theory by special relativity in their clearest form. We show here that in this case the ergodic theorem can appear to fail in the sense that a system ergodic in I_0 is not ergodic in I.

The ensemble of N_0 systems (say) which are strictly at rest in I_0 yield certain probabilities $Q_{j0} = n_{j0}/N_0$ for the states j of the system, n_{j0} being the number of systems in state j. For an observer in an inertial frame I this ratio will be the same, except possibly for a permutation of the labels j. Hence there exists a labelling of states j such that

$$Q_{j0} = Q_j. \tag{1.1}$$

It remains to show that the time-based probabilities violate this invariance.

In § 2 we obtain a Lorentz transformation of time-based probabilities. In § 3 we obtain expected (and clearly correct) results from this transformation. For clarity of exposition we deal with a special class of simple systems as specified in (ii) of § 2. We shall use the concepts of confined and inclusive systems, as introduced earlier (Landsberg and Johns 1967).

2. Basic results for time-based probabilities

We enumerate the concepts required over and above those needed in non-relativistic mechanics:

(i) In each of the (enumerable) states i of a system A, it is free of all external interactions and thus possesses a definite energy–momentum four-vector $E_i^\mu = (c\boldsymbol{P}_i, E_i)$ for each inertial frame I. Interactions within the system which leave E_i^μ unchanged can also bring about a change from state i.

(ii) Interaction events, whether internal or external, can occur, and change one state (say i) of A to another state (say j). These events are assumed to be separated by time-like intervals which define times s_i in I for which A is in the state i.

The assumption in (ii) ensures that the time-ordering of states is the same for all inertial frames. This would not be the case if some of the events j were separated by space-like intervals. The assumption (ii) is certainly fulfilled for a gas of one particle in a box. The results obtained from it also hold for any number of non-interacting particles in a box, since this can be considered to be simply a superposition of several of such one-particle systems. Once interactions are allowed, the usual difficulties of relativistic mechanics (Havas 1965) arise. Such situations are not considered here.

(iii) Since the system (in state i) moves with constant velocity $\boldsymbol{u}_i (= c^2\boldsymbol{P}_i/E_i)$, there is a flow of momentum in the direction of its motion at a rate $\boldsymbol{u}_i \,.\, \boldsymbol{P}_i$. Now if $\boldsymbol{u}_i$ is the same for all i (for example if the system is without external interactions), then the system conveys no momentum in the frame I_0 in which it is permanently at rest ($\boldsymbol{u}_{i0} = \boldsymbol{0}$). On the other hand, if $\boldsymbol{u}_i$ varies with i, there is an external interaction and an overall flow of momentum in every frame, even in the frame I_0 (which will always be taken to be the frame in which the system is on *average* at rest). This situation would arise in a confined system in which the pressure exerted by the container is the external interaction which produces the flow of momentum.

(iv) All times s_i for which the system is in state i during a long period of observation τ in I are summed to yield a total time t_i for state i. The time-based probabilities for states i in frame I are now defined to be

$$\Pi_i = t_i/\tau \tag{2.1}$$

so that

$$\sum_i \Pi_i = 1.$$

Consider now frame I_0 in standard configuration with frame I (see figure 1), so that with $\gamma \equiv \{1-(w/c)^2\}^{-1/2}$

$$\tau = \sum_i t_i = \gamma \sum_i (t_{i0} + \boldsymbol{w} \,.\, \boldsymbol{x}_{i0}/c^2). \tag{2.2}$$

The space and time separations in I_0 between the events which define the space–time interval for which the system is in state i have been denoted by $\boldsymbol{x}_{i0}$ and t_{i0}. It is important that every such interval be time-like rather than space-like; otherwise t_i, and hence Π_i, can appear negative for some frames of reference. In other words, the

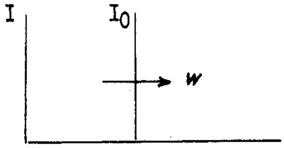

Figure 1.

separations must be determined by the motion of some point representative of the system as a whole (for example, the centre of gravity defined for any state i, and a selected frame I or I_0), and not by the microscopic motion of an arbitrary particle. If this precaution is not taken, then even for a system of only two particles, x_{i0}/t_{i0} may exceed c if the two terminal events are nearly simultaneous collisions of the particles with different points on the container wall. It is because attention is confined here to systems satisfying condition (ii) that one can put

$$\boldsymbol{x}_{i0} = \boldsymbol{u}_{i0} t_{i0} \tag{2.3}$$

where $\boldsymbol{u}_{i0}$ (in general non-zero although its time average *is* zero) is the velocity of the system.

It follows from (2.2) and (2.3) that

$$\tau = \gamma \tau_0; \tag{2.4}$$

since the total displacement in I_0 is, by definition, zero

$$\sum_i \boldsymbol{x}_{i0} = \sum_i \boldsymbol{u}_{i0} t_{i0} = \tau_0 \sum_i \boldsymbol{u}_{i0}\, \Pi_{i0} = \tau_0 c^2 \sum_i \frac{\boldsymbol{P}_{i0}}{E_{i0}}\, \Pi_{i0} = \boldsymbol{0}. \tag{2.5}$$

Also

$$\frac{\Pi_i}{\Pi_{i0}} = \frac{t_i}{\gamma t_{i0}} = 1 + \frac{\boldsymbol{w} \,.\, \boldsymbol{P}_{i0}}{E_{i0}} = \frac{E_i}{\gamma E_{i0}} = \frac{1}{\gamma^2(1 - \boldsymbol{w} \,.\, \boldsymbol{P}_i/E_i)}. \tag{2.6}$$

This is the *Lorentz transformation of time-based probabilities.*

The significance of (2.6) can be understood physically. Consider a state i in which $\boldsymbol{P}_{i0}$ has a component parallel to $\boldsymbol{w}$. More time is spent in this state when judged from frame I than when judged from I_0. The reason is that, considering a confined system, it is clearly moving with a component of its velocity parallel to the velocity of the box. Why equation (2.6) does not apply directly to an inclusive system is discussed at the end of § 3. Conversely, if $\boldsymbol{P}_{i0}$ has a component antiparallel to $\boldsymbol{w}$ the time for state i is shortened by going from I_0 to I. Thus the whole distribution Π_{i0} is displaced in the direction of $\boldsymbol{w}$. We have already utilized this effect in a slightly different context (Landsberg and Johns 1967, § 3).

3. Main consequences of the Lorentz transformation of time-based probabilities

(i) In frame I, the mean velocity of the system is $\boldsymbol{w}$ (the velocity in I of I_0). This can be demonstrated for the components of $\boldsymbol{u}_i$ parallel to $\boldsymbol{w}$, $(\boldsymbol{u}_{i\parallel})$, and perpendicular to $\boldsymbol{w}$, $(\boldsymbol{u}_{i\perp})$; thus by use of (2.5) and (2.6)

$$\sum_i \boldsymbol{u}_{i\parallel}\,\Pi_i = \sum_i \frac{(\boldsymbol{u}_{i0\parallel}+\boldsymbol{w})}{(1+\boldsymbol{w}\,.\,\boldsymbol{u}_{i0}/c^2)}\left(1+\frac{\boldsymbol{w}\,.\,\boldsymbol{u}_{i0}}{c^2}\right)\Pi_{i0}$$

$$= \sum_i (\boldsymbol{u}_{i0\parallel}+\boldsymbol{w})\,\Pi_{i0} = \boldsymbol{w} \tag{3.1}$$

and

$$\sum_i \boldsymbol{u}_{i\perp}\,\Pi_i = \sum_i \frac{\boldsymbol{u}_{i0\perp}}{\gamma(1+\boldsymbol{w}\,.\,\boldsymbol{u}_{i0}/c^2)}\left(1+\frac{\boldsymbol{w}\,.\,\boldsymbol{u}_{i0}}{c^2}\right)\Pi_{i0}$$

$$= \frac{1}{\gamma}\sum_i \boldsymbol{u}_{i0\perp}\,\Pi_{i0} = \boldsymbol{0}. \tag{3.2}$$

(ii) The mean values of the energy and momentum in a general frame of reference (such as I) are expressed by

$$\langle E\rangle = \sum_i \Pi_i E_i \qquad \text{and} \qquad \langle \boldsymbol{P}\rangle = \sum_i \Pi_i \boldsymbol{P}_i. \tag{3.3}$$

These can be rewritten in terms of the quantities of I_0 by

$$\langle E\rangle = \gamma\sum_i \Pi_{i0}\left(1+\frac{\boldsymbol{w}\,.\,\boldsymbol{P}_{i0}}{E_{i0}}\right)(E_{i0}+\boldsymbol{w}\,.\,\boldsymbol{P}_{i0})$$

$$= \gamma\sum_i \Pi_{i0}\left\{E_{i0}+2\boldsymbol{w}\,.\,\boldsymbol{P}_{i0}+\frac{1}{c^2}(\boldsymbol{w}\,.\,\boldsymbol{P}_{i0})(\boldsymbol{w}\,.\,\boldsymbol{u}_{i0})\right\}. \tag{3.4}$$

Also

$$\langle \boldsymbol{P}\rangle = \gamma\sum_i \Pi_{i0}\left(1+\frac{\boldsymbol{w}\,.\,\boldsymbol{P}_{i0}}{E_{i0}}\right)\left(\boldsymbol{P}_{i0}+\frac{\boldsymbol{w}}{c^2}E_{i0}\right)$$

$$= \gamma\sum_i \Pi_{i0}\left\{\boldsymbol{P}_{i0}+\frac{\boldsymbol{w}}{c^2}(\boldsymbol{w}\,.\,\boldsymbol{P}_{i0})+\frac{\boldsymbol{w}}{c^2}E_{i0}+\frac{\boldsymbol{P}_{i0}}{c^2}(\boldsymbol{w}\,.\,\boldsymbol{u}_{i0})\right\}. \tag{3.5}$$

(iii) The interaction with the walls will normally introduce a fluctuation in the energy and momentum of the system. Writing $\langle\ldots\rangle_0$ for an averaging procedure in which Π_{i0} is used, we shall restrict the systems to be considered by the following condition, which is incidentally satisfied, in virtue of (2.5), if E_{i0} is the same for all i, namely

$$\langle \boldsymbol{P}_0\rangle_0 \equiv \sum_i \Pi_{i0}\boldsymbol{P}_{i0} = 0. \tag{3.6}$$

We also put

$$\langle E_0\rangle_0 = \sum_i \Pi_{i0}E_{i0}. \tag{3.7}$$

The results (3.4), (3.5) are therefore

$$\langle E \rangle = \gamma\{\langle E_0 \rangle_0 + \sum_i \Pi_{i0}(\boldsymbol{w} \,.\, \boldsymbol{P}_{i0})(\boldsymbol{w} \,.\, \boldsymbol{u}_{i0})/c^2\} \tag{3.8}$$

$$\langle \boldsymbol{P} \rangle = (\gamma/c^2)\{\langle E_0 \rangle_0 \boldsymbol{w} + \sum_i (\boldsymbol{w} \,.\, \boldsymbol{u}_{i0})\, \Pi_{i0} \boldsymbol{P}_{i0}\}. \tag{3.9}$$

(iv) In order to simplify these equations further the following result with be established for an equilibrium system under external pressure p:

$$pV_0\hat{\boldsymbol{n}} = \sum_i (\hat{\boldsymbol{n}} \,.\, \boldsymbol{u}_{i0})\, \Pi_{i0} \boldsymbol{P}_{i0} \tag{3.10}$$

where $\hat{\boldsymbol{n}}$ is any unit vector. The argument, which is related to the virial theorem, is as follows. Consider a surface element with area δa and unit normal $\hat{\boldsymbol{n}}$ fixed at rest in I_0 within the volume occupied by the system. From the macroscopic point of view the total momentum flowing across this surface in the $\hat{\boldsymbol{n}}$ direction in time δt will be

$$p\, \delta a\, \delta t\, \hat{\boldsymbol{n}}.$$

Viewed statistically, the system as a whole moves back and forth in I_0 with various velocities $\boldsymbol{u}_{i0}$, whilst the surface element which we consider always remains at rest in I_0. Thus the volume of that part of the moving system which passes through δa in time δt is $\delta V_0 \equiv \delta a\, \hat{\boldsymbol{n}} \,.\, \boldsymbol{u}_{i0} \delta t$. Since the spatial distribution of the momentum is completely random within the system, the mean momentum passing through δa in this time will be

$$\frac{\delta V_0}{V_0} \boldsymbol{P}_{i0} = \frac{\delta a(\hat{\boldsymbol{n}} \,.\, \boldsymbol{u}_{i0})\delta t \boldsymbol{P}_{i0}}{V_0}.$$

The mean value over all states i of the rate of flow of momentum across δa in time δt is therefore

$$\sum_i \Pi_{i0} \frac{\delta a(\hat{\boldsymbol{n}} \,.\, \boldsymbol{u}_{i0})\delta t \boldsymbol{P}_{i0}}{V_0} = \frac{\delta a\, \delta t}{V_0} \sum_i \Pi_{i0} \boldsymbol{P}_{i0}(\hat{\boldsymbol{n}} \,.\, \boldsymbol{u}_{i0}).$$

Equating this with the previous result for this quantity, and cancelling $\delta a\, \delta t$, one obtains (3.10).

(v) The sum in (3.8), using (3.10), becomes with the choice of $\hat{\boldsymbol{n}}$ given by $\boldsymbol{w} = w\hat{\boldsymbol{n}}$

$$c^{-2}w^2\hat{\boldsymbol{n}} \,.\, \{\sum_i (\hat{\boldsymbol{n}} \,.\, \boldsymbol{u}_{i0})\, \Pi_{i0} \boldsymbol{P}_{i0}\} = (w^2/c^2)pV_0.$$

Similarly, the sum in (3.9) is

$$w \sum_i (\hat{\boldsymbol{n}} \,.\, \boldsymbol{u}_{i0})\, \Pi_{i0} \boldsymbol{P}_{i0} = \boldsymbol{w} p V_0.$$

Hence the standard results

$$\langle E \rangle = \gamma\{\langle E_0 \rangle_0 + (w/c)^2 pV_0\} \tag{3.11}$$

$$\langle \boldsymbol{P} \rangle = \gamma(\langle E_0 \rangle_0 + \gamma p V_0)(\boldsymbol{w}/c^2) \tag{3.12}$$

are found. This acts as a confirmation of the general approach given here to time-based probabilities.

It is clear from (3.10) that if $\boldsymbol{u}_{i0} = \boldsymbol{0}$ for all states i, then $p = 0$. In this case (3.11) and (3.12) reduce to the *Lorentz transformation* of the energy–momentum four-vector as is appropriate for a *free* or *inclusive* system. For the former of these, the lack of any external interaction can be considered to be responsible for the absence of change in its velocity. However, it would be a mistake to apply such an interpretation also to an inclusive system (which, it will be recalled, is formed of two parts, namely the container and the confined system within it). The state of the inclusive system is determined by the state of this confined system rather than by any hypothesized behaviour of the container. It is, therefore, the velocity of the confined system which must be used in the transformation (2.6) when it is applied to an inclusive system. Since the terms in (3.11) and (3.12) are derived in part from the transformation (2.6), which cannot contain the variables of an inclusive system, it is clear that these equations cannot represent the Lorentz transformation of the energy and momentum of an inclusive system. It is thus only for a free system that the result $\boldsymbol{u}_{i0} = \mathbf{0}$ may be combined with equation (2.6) to give

$$\Pi_i = \Pi_{i0}.$$

Conventional ensembles as envisaged in (1.1) (i.e. ensembles whose systems are permanently at rest in a certain inertial frame) and time-based probabilities for free systems thus lead to Lorentz-invariant ergodicity. It is, however, the pressurized (inclusive or confined) system which represents the important case, and this requires further discussion (see § 5 and Johns and Landsberg 1970).

The circumstance that the Π_i transform yields the two additional terms in each of (3.4) and (3.5). One of these reduces to zero in virtue of (3.6). The other (the last term in each of (3.4) and (3.5)) gives rise to the pV term, characteristic of confined systems. Thus the correct formulae (3.11) and (3.12) are given by the above treatment for confined systems *only if the probability transforms.*

4. Comparison with a classical argument

In the preceding investigations it has been shown that, for the cases of one-particle systems and equilibrium systems of non-interacting particles, the ergodic theorem can only be said to hold if each observer in a different inertial frame is allowed to establish his own ensemble for the purpose of describing the system. The immediate reason for this is seen to be the Lorentz transformation of the time-based probabilities applicable to each of the discrete energy–momentum states i in which the system can exist.

This transformation is in accordance with the strictly classical behaviour of the particle density f in phase space (μ space) which is defined

$$f \equiv \frac{\delta n}{\delta \omega}. \tag{4.1}$$

Here n is the expected number of particles within a phase-space element of volume $d\omega$. The fact that f is Lorentz invariant is an indication that δn and $\delta \omega$ are increased by the same factor under a Lorentz transformation. This may be proved as follows.

Firstly the phase-space element may be defined by the canonical coordinates of position and momentum in frame I_0. Thus its phase volume in I_0 is given by

$$\delta\omega_0 = \delta p_{10}\delta p_{20}\delta p_{30}\delta x_{10}\delta x_{20}\delta x_{30}.$$

Under a Lorentz transformation of $\delta\omega_0$ to frame I, the momentum coordinates contribute to the transformation a factor equal to the Jacobian determinant:

$$\left|\frac{\partial p_j}{\partial p_{k0}}\right| = \frac{\epsilon}{\epsilon_0} \qquad (j, k = 1, 2, 3)$$

where ϵ, ϵ_0 are the energies of particles with momenta $\boldsymbol{p}$, $\boldsymbol{p}_0$. The position coordinates for the element, specified to be at rest in I_0, introduce a Lorentz contraction factor $1/\gamma$. Thus

$$\delta\omega = \frac{\epsilon}{\gamma\epsilon_0}\delta\omega_0. \tag{4.2}$$

Secondly, the number δn of particles in $\delta\omega$ (measured in frame I) depends on the time which each particle spends in $\delta\omega$. Hence δn transforms in a way which depends on the velocity $\boldsymbol{u} = \boldsymbol{p}c^2/\epsilon$, which is possessed by *every* particle in $\delta\omega$ (except for variations of small magnitude δp). Thus if the proper time (i.e. the time in its own inertial frame) spent by a particle in $\delta\omega$ is $\delta\tau$, the measurement in frame I of this time will be

$$\delta t = \delta\tau\left(1-\frac{u^2}{c^2}\right)^{-1/2} = \delta\tau\,\gamma\left(1-\frac{u_0^2}{c^2}\right)^{-1/2}\left(1+\frac{\boldsymbol{u}_0\cdot\boldsymbol{w}}{c^2}\right).$$

Similarly, the time measured in I_0 is

$$\delta t_0 = \delta\tau\left(1-\frac{u_0^2}{c^2}\right)^{-1/2}.$$

The ratio $\delta t/\delta t_0$ is therefore given by

$$\frac{\delta t}{\delta t_0} = \gamma\left(1+\frac{\boldsymbol{u}_0\cdot\boldsymbol{w}}{c^2}\right) = \frac{\epsilon}{\epsilon_0}.$$

The overall rate at which particles enter $\delta\omega$ is slower in I than in I_0 by a factor $1/\gamma$, owing to the Lorentz time dilation applied to the element $d\omega$ at rest in I_0. This factor also affects the number of particles in the system at any time. Thus

$$\delta n = \frac{\delta t}{\delta t_0}\frac{1}{\gamma}\delta n_0$$

$$= \frac{\epsilon}{\gamma\epsilon_0}\delta n_0. \tag{4.3}$$

The Lorentz invariance of the quantity f, as defined by (4.1), follows at once from (4.2) and (4.3). Of particular consequence here is (4.3), which gives the expected number of particles in a phase-space element in frame I regardless of the phase volume of that element in I. It is the classical statistical mechanical analogue of the probability Π_i (equation (2.6)) of a system being in a *discrete* state, when no question of phase volume arises. Just as (3.11, 3.12) were derived from (2.6), so these equations can also be derived from (4.3). However, the Σ in § 3 would have to be replaced by an integral over $d\omega$ with the distribution function f as a weighting factor. It is this argument only which appears already in the literature (Pathria 1966, Møller 1968, van Kampen 1969). The result derived here in §§ 2 and 3 is therefore an extension of the classical result.

Since the velocities $\boldsymbol{u}$ and $\boldsymbol{u}_0$ in this section are particle velocities rather than system velocities, there is no question of an inclusive system being described here. Instead, if $\boldsymbol{u}_0 = \boldsymbol{0}$ in all regions of phase space occupied by the particles, we merely have the unrealistic situation of a system of completely motionless particles.

5. Conclusions

Suppose a given system S is described from an arbitrary inertial frame I, and is represented by an ensemble E_0. We have introduced an inertial frame I_0 in which S is assumed to be *on average* at rest, while the systems constituting E_0 are assumed, in accordance with a usual convention, to be *strictly* at rest. Then

$\boldsymbol{u}_{i0} = c^2\boldsymbol{P}_{i0}/E_{i0} = \mathbf{0}$ for the systems of E_0, but $\boldsymbol{u}_{i0}$ can be non-zero for S. Equations (1.1) and (2.6) now show that one will find, in general, a lack of equality between time and ensemble-based probabilities in I, $\Pi_i \neq Q_i$, even if $\Pi_{i0} = Q_{i0}$. Ergodicity is then not a Lorentz-invariant concept. We replace this convention picture of ensembles, however, and instead require the restrictions on the motion of the system S to be the same as those imposed on the systems of E_0. Either they are *all* strictly at rest in I_0, or they are *all* constrained to be only on average at rest in I_0. Here 'rest' or 'motion' can refer to the container (the 'black box') in the case of an inclusive system, and to the centre of mass in the case of a confined system. The two cases $\boldsymbol{u}_{i0} = \mathbf{0}$, $\langle \boldsymbol{u}_{i0} \rangle_0 = \mathbf{0}$ will be discussed in turn, it being noted that the transformation properties of the probabilities (2.6) must be the same for confined and inclusive systems (see § 3).

(A) $\boldsymbol{u}_{i0} = \mathbf{0}$. If all quantities are interpreted as applying to a confined system, the pressure is zero by (3.10) and the transformation formulae (3.11) and (3.12) hold then only for the trivial case of a confined system at zero pressure. By equation (2.6) one finds Lorentz-invariant probabilities Π_i and a Lorentz-invariant notion of ergodicity.

(B) $\langle \boldsymbol{u}_{i0} \rangle_0 = \mathbf{0}$. Apart from differences due to different initial states, and the like, each system of the ensemble E_0 has statistically the same kind of behaviour in time as the system S. This was investigated in § 2. An army of observers, all at rest in a frame I, make observations simultaneously in I on the systems of E_0. These observations are not simultaneous in I_0. From repeated sets of observations of this type the ensemble-based probabilities Q_i are derived. However, the time-based probability distribution of each system of the ensemble is displaced, as described at the end of § 2, in passing from I_0 to I. As a consequence the Q's will be displaced in just the way in which the Π's are displaced. Thus ergodicity is a Lorentz-invariant notion also in this case.

This conclusion depends, however, on an interpretation of ensembles which allows the individual systems to be in motion, while restricting them to be on average at rest in the frame in which the system S is also on average at rest.

The main results of this paper are seen to be:

(i) The discrete probabilities of statistical mechanics are not Lorentz invariant (§§ 2 and 3), a property which they share with the continuous classical distribution functions δn (§ 4).

(ii) None the less, with a suitable interpretation concerning the notion of an ensemble the property of ergodicity can be arranged to be Lorentz invariant.

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